

Approximate amenability of Schatten classes, Lipschitz algebras and second duals of Fourier algebras

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Abstract

Amenability of any of the algebras described in the title is known to force them to be finite-dimensional. The analogous problems for *approximate* amenability have been open for some years now. In this article we give a complete solution for the first two classes, using a new criterion for showing that certain Banach algebras without bounded approximate identities cannot be approximately amenable. The method also provides a unified approach to existing non-approximate amenability results, and is applied to the study of certain commutative Segal algebras.

Using different techniques, we prove that *bounded* approximate amenability of the second dual of a Fourier algebra implies that it is finite-dimensional. Some other results for related algebras are obtained.

MSC 2000: 46H20 (primary), 43A20 (secondary).

1 Introduction

The concepts of approximate amenability, approximate contractibility and essential amenability, and some other related concepts, were introduced and studied in [8] and further developed in [11, 12, 4, 10, 2]. The studies include determining when various classes of Banach algebras are, or are not, approximately amenable. Attempting to prove that a given Banach algebra is *not* approximately amenable can be much more difficult than for ‘classical’ amenability.

It was shown in [4] that for $1 \leq p < \infty$ the algebras ℓ^p , equipped with pointwise multiplication, are not approximately amenable; this has motivated the question of whether the Schatten classes $S^p(H)$ are approximately amenable. In [10, §7], partial results were also obtained about approximate amenability, or otherwise, of the little Lipschitz algebras $\text{lip}_\alpha(X)$.

It remains an open question, to the authors’ knowledge, if an approximately amenable Banach algebra \mathcal{A} must necessarily have a bounded approximate identity. The authors

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have recently proved, in collaboration with Y. Zhang, that the *a priori* stronger notion of bounded approximate contractibility forces the existence of a bounded approximate identity; see [2, §3]. The main result of Section 2 gives a criterion for proving that certain algebras without bounded approximate identities cannot be approximately amenable.

For example, we use our criterion to show that neither Schatten classes on infinite-dimensional Hilbert spaces, nor Lipschitz algebras on infinite compact metric spaces, are approximately amenable. Another question left open from previous studies is whether every proper Segal subalgebra of the group algebra of a locally compact group fails to be approximately amenable. We prove that this is the case for the group \mathbb{R}^n . We correct an error in the literature on approximate amenability of the semigroup algebras of Brandt semigroups, and also use our criterion to provide new proofs for several results obtained by other authors.

Another open question has been whether every approximately amenable Banach algebra whose underlying space is reflexive, as a Banach space, is finite-dimensional. We show this is the case, under the additional hypotheses of commutativity and *bounded* approximate amenability. Finally we show that the second dual algebra of the Fourier algebra of a locally compact group is boundedly approximately amenable, if and only if the group is finite. This extends the result in the amenable case, which was proved by A. T. Lau and R. J. Loy [21] and independently by E. Granirer [13].

Now we recall some basic definitions. A derivation D from a Banach algebra \mathcal{A} into a Banach \mathcal{A} -bimodule X is **approximately inner**, if there exists a net (ad_{x_i}) of inner derivations from \mathcal{A} into X such that D is the strong-operator-topology limit of (ad_{x_i}) . The Banach algebra \mathcal{A} is **approximately amenable** if every continuous derivation D from \mathcal{A} into any dual Banach \mathcal{A} -bimodule X^* is approximately inner, for all Banach \mathcal{A} -bimodules X . If in the above definition the nets of inner derivations can always be taken to be bounded, then \mathcal{A} is **boundedly approximately amenable**.

Approximate amenability and bounded approximate amenability can be characterized by the existence of so-called “approximate diagonals” ([8, 10]), and this will be used several times in the proofs of our main results.

2 A criterion for ruling out approximate amenability

It was shown in [2, §4] that the Fourier algebra of the free group on two generators is not (operator) approximately amenable. Motivated by some of the ideas in that argument, we can now give a criterion which shows that numerous examples of Banach algebras without bounded approximate identities fail to be approximately amenable.

In particular, we have the following result, which resolves a question that had been open since [8]. (Recall from [8, Corollary 7.1] that $\mathcal{S}_p(H)$ is *essentially amenable*.)

THEOREM 2.1. *Let $1 \leq p < \infty$ and let $\mathcal{S}_p(H)$ denote the ideal of Schatten class operators on an infinite-dimensional Hilbert space H . Then $\mathcal{S}_p(H)$ is not approximately amenable.*

We shall deduce this from a much more general result, whose statement requires some technical definitions. Given a Banach algebra \mathcal{A} , we define the **multiplier norm on \mathcal{A}** by $\|a\|_{\text{mul}} := \max(\|\lambda_a\|, \|\rho_a\|)$, where $\lambda_a : \mathcal{A} \rightarrow \mathcal{A}$, $x \mapsto ax$ is left multiplication by a and $\rho_a : \mathcal{A} \rightarrow \mathcal{A}$, $x \mapsto xa$ is right multiplication by a .

DEFINITION 2.2. A **separated, unbounded, multiplier-bounded configuration** (or **SUM configuration** for short) in \mathcal{A} consists of two sequences $(u_n), (p_n) \subseteq \mathcal{A}$ which satisfy the following properties.

(Separated) $u_n p_n = p_n u_n = u_n$ for all n ; and $u_j p_k = p_k u_j = 0$, whenever $j \neq k$.

(Unbounded) $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$.

(Multiplier-bounded) The sequences $(\|u_n\|_{\text{mul}})$ and $(\|p_n\|_{\text{mul}})$ are bounded.

REMARKS 2.3.

- (i) In the definition of a SUM configuration, by passing to a subsequence if necessary, we may assume that the sequence $(\|u_n\|)_{n \geq 1}$ grows faster than any prescribed sequence of positive reals.
- (ii) If \mathcal{A} is a Banach algebra with a bounded approximate identity, then $\|\cdot\|$ and $\|\cdot\|_{\text{mul}}$ are equivalent norms, by Cohen's factorization theorem, so that \mathcal{A} contains no SUM configurations.
- (iii) In several, though not all, of the applications below, one can choose the (u_n) to be pairwise orthogonal idempotents, and take $p_n = u_n$.

The definition of a SUM configuration may seem overly technical. It is therefore convenient to have a simpler criterion that ensures the existence of such sequences.

LEMMA 2.4. *Let \mathcal{A} be a Banach algebra. Suppose that there exists an unbounded but multiplier-bounded sequence $(E_n)_{n \geq 1} \subseteq \mathcal{A}$, such that $E_n E_{n+1} = E_n = E_{n+1} E_n$ for all n . Then \mathcal{A} contains a SUM configuration.*

Proof of Lemma 2.4. Note that, for $1 \leq j < k$, we have $E_j E_k = E_j = E_k E_j$. Moreover, by passing to a subsequence if necessary, we may assume without loss of generality that $\|E_{n+1}\| > \|E_n\|$ for all n and that $\|E_{n+1}\| - \|E_n\| \rightarrow \infty$ as $n \rightarrow \infty$.

We now define

$$u_n := E_{3n} - E_{3n-1} \quad , \quad p_n := E_{3n+1} - E_{3n-2} \quad \text{for } n \geq 1.$$

Clearly both $(\|u_n\|_{\text{mul}})$ and $(\|p_n\|_{\text{mul}})$ are uniformly bounded sequences, since the sequence (E_n) is multiplier-bounded. Moreover, our construction ensures that

$$\|u_n\| \geq \|E_{3n}\| - \|E_{3n-1}\| \rightarrow \infty.$$

It remains to verify the ‘separatedness’ conditions. Since the set $\{E_n : n \in \mathbb{N}\}$ is commutative, u_j and p_k commute for every j, k . We have

$$u_n p_n = p_n u_n = (E_{3n+1} - E_{3n-2})(E_{3n} - E_{3n-1}) = E_{3n} - E_{3n-1} = u_n \quad \text{for all } n.$$

If $j < k$, then, since $3k - 2 > 3j$,

$$u_j p_k = p_k u_j = (E_{3k+1} - E_{3k-2})(E_{3j} - E_{3j-1}) = 0;$$

and similarly, since $3k - 1 > 3j + 1$,

$$u_k p_j = p_j u_k = (E_{3j+1} - E_{3j-2})(E_{3k} - E_{3k-1}) = 0.$$

Thus (u_n) and (p_n) form a SUM configuration, as required. \square

The point of our definition is the following result, whose proof should be compared with the main arguments in [2, §4].

THEOREM 2.5. *Let \mathcal{A} be a Banach algebra which contains a SUM configuration. Then \mathcal{A} is not approximately amenable.*

Proof of Theorem 2.5. Let $(u_n), (p_n)$ be a SUM configuration in \mathcal{A} , and put

$$C := \max\{\sup_n \|u_n\|_{\text{mul}}, \sup_n \|p_n\|_{\text{mul}}\}.$$

Elementary calculations using the definition of the projective tensor norm show that for any $w \in \mathcal{A} \hat{\otimes} \mathcal{A}$,

$$\max(\|u_n \cdot w\|, \|w \cdot u_n\|, \|p_n \cdot w\|, \|w \cdot p_n\|) \leq C \|w\| \quad \text{for all } n. \quad (*)$$

By passing to a subsequence if necessary, we can without loss of generality assume that in addition to satisfying $(*)$, our sequence (u_n) satisfies

$$\|u_n\| \geq (n+1)^4 \quad \text{for all } n. \quad (+)$$

(The exact rate of growth is not important, but we need it to be moderately fast.)

Note that the ‘separatedness conditions’ imply that the (u_n) are pairwise orthogonal, since for $j \neq k$ we have $u_j u_k = u_j p_j u_k = 0$. Define $a, b \in \mathcal{A}$ by

$$a = \sum_{j \geq 1} (2j-1)^{-2} \|u_{2j-1}\|^{-1} u_{2j-1} \quad \text{and} \quad b = \sum_{k \geq 1} (2k)^{-2} \|u_{2k}\|^{-1} u_{2k},$$

and observe that for each $n \in \mathbb{N}$:

$$\begin{aligned} p_{2n-1} a &= (2n-1)^{-2} \|u_{2n-1}\|^{-1} u_{2n-1}; & a u_{2n} &= 0; \\ b p_{2n} &= (2n)^{-2} \|u_{2n}\|^{-1} u_{2n}; & u_{2n-1} b &= 0. \end{aligned} \quad (**)$$

Our proof now proceeds by contradiction. Suppose that \mathcal{A} is approximately amenable, and let $\varepsilon > 0$. Then, since approximate amenability and approximate contractibility are the same, there exist $M \in \widehat{\mathcal{A} \otimes \mathcal{A}}$ and $F, G \in \mathcal{A}$ such that:

$$\|a \cdot M - M \cdot a - a \otimes G + F \otimes a\| < \varepsilon; \quad (2.1a)$$

$$\|b \cdot M - M \cdot b - b \otimes G + F \otimes b\| < \varepsilon; \quad (2.1b)$$

$$\|a - aF\| < \varepsilon. \quad (2.2)$$

From (2.1a) and the observations (*) and (**), we see that for any $j, k \in \mathbb{N}$,

$$\begin{aligned} C^2\varepsilon &> \|p_{2j-1} \cdot (a \cdot M - M \cdot a - a \otimes G + F \otimes a) \cdot u_{2k}\| \\ &= \|p_{2j-1}a \cdot M \cdot u_{2k} - p_{2j-1}a \otimes Gu_{2k}\| \\ &= \|(2j-1)^{-2}\|u_{2j-1}\|^{-1}(u_{2j-1} \cdot M \cdot u_{2k} - u_{2j-1} \otimes Gu_{2k}), \end{aligned}$$

so that

$$\frac{\|u_{2j-1} \cdot M \cdot u_{2k} - u_{2j-1} \otimes Gu_{2k}\|}{\|u_{2j-1}\| \|u_{2k}\|} \leq C^2\varepsilon \frac{(2j-1)^2}{\|u_{2k}\|}. \quad (2.3a)$$

A similar argument, using (2.1b) instead of (2.1a), yields

$$\frac{\|-u_{2j-1} \cdot M \cdot u_{2k} + u_{2j-1}F \otimes u_{2k}\|}{\|u_{2j-1}\| \|u_{2k}\|} \leq C^2\varepsilon \frac{(2k)^2}{\|u_{2j-1}\|}; \quad (2.3b)$$

combining (2.3a) and (2.3b) using the triangle inequality then gives

$$\begin{aligned} C^2\varepsilon \left(\frac{(2j-1)^2}{\|u_{2k}\|} + \frac{(2k)^2}{\|u_{2j-1}\|} \right) &\geq \left\| \frac{u_{2j-1}F \otimes u_{2k}}{\|u_{2j-1}\| \|u_{2k}\|} - \frac{u_{2j-1} \otimes Gu_{2k}}{\|u_{2j-1}\| \|u_{2k}\|} \right\| \\ &\geq \left| \frac{\|u_{2j-1}F\|}{\|u_{2j-1}\|} - \frac{\|Gu_{2k}\|}{\|u_{2k}\|} \right| \end{aligned}$$

(Here we used the inequality $\|x - y\| \geq |\|x\| - \|y\||$, valid in any normed space, together with the fact that the projective tensor norm is a cross-norm). Using the growth condition (+), we therefore have

$$\left| \frac{\|u_{2j-1}F\|}{\|u_{2j-1}\|} - \frac{\|Gu_{2k}\|}{\|u_{2k}\|} \right| \leq C^2\varepsilon \left(\frac{(2j-1)^2}{(2k+1)^4} + \frac{(2k)^2}{(2j)^4} \right). \quad (2.4)$$

Taking $j = k$ in (2.4) gives

$$\left| \frac{\|u_{2k-1}F\|}{\|u_{2k-1}\|} - \frac{\|Gu_{2k}\|}{\|u_{2k}\|} \right| \leq C^2\varepsilon \left(\frac{(2k-1)^2}{(2k+1)^4} + \frac{(2k)^2}{(2k)^4} \right) \leq \frac{2C^2}{(2k)^2}\varepsilon,$$

while taking $j = k+1$ in (2.4) gives

$$\left| \frac{\|u_{2k+1}F\|}{\|u_{2k+1}\|} - \frac{\|Gu_{2k}\|}{\|u_{2k}\|} \right| \leq C^2\varepsilon \left(\frac{(2k+1)^2}{(2k+1)^4} + \frac{(2k)^2}{(2k+2)^4} \right) \leq \frac{2C^2}{(2k+1)^2}\varepsilon.$$

Combining the preceding two inequalities, we see that, for every $k \geq 1$,

$$\left| \frac{\|u_{2k-1}F\|}{\|u_{2k-1}\|} - \frac{\|u_{2k+1}F\|}{\|u_{2k+1}\|} \right| \leq 2C^2\varepsilon \left(\frac{1}{(2k)^2} + \frac{1}{(2k+1)^2} \right).$$

Then, since $\lim_{m \rightarrow \infty} \|u_{2m+1}\|^{-1} \|u_{2m+1}F\| = 0$, we deduce that

$$\begin{aligned} \frac{\|u_1F\|}{\|F\|} &= \lim_{m \rightarrow \infty} \left| \frac{\|u_1F\|}{\|u_1\|} - \frac{\|u_{2m+1}F\|}{\|u_{2m+1}\|} \right| \\ &= \lim_{m \rightarrow \infty} \left| \sum_{k=1}^m \frac{\|u_{2k-1}F\|}{\|u_{2k-1}\|} - \frac{\|u_{2k+1}F\|}{\|u_{2k+1}\|} \right| \\ &\leq 2C^2\varepsilon \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \frac{1}{(2k+1)^2} \leq 2C^2\varepsilon. \end{aligned} \tag{2.5}$$

However: going back to the ‘approximate identity’ formula (2.2), we also have

$$C\varepsilon \geq \|u_1(a - aF)\| = \left\| \frac{u_1}{\|u_1\|} - \frac{u_1F}{\|u_1\|} \right\| \geq \left| 1 - \frac{\|u_1F\|}{\|u_1\|} \right|;$$

combining this with (2.5), we conclude that

$$1 \leq C\varepsilon + 2C^2\varepsilon.$$

Since ε was arbitrarily small this yields a contradiction, and therefore \mathcal{A} is not approximately amenable. \square

Proof of Theorem 2.1. We can now complete the proof that $\mathcal{S}_p(H)$ is not approximately amenable. While we could use Lemma 2.4, it is more instructive to construct an explicit SUM configuration as follows. Fix an infinite orthonormal set $(\delta_n)_{n \geq 1}$ and take P_1 to be the orthogonal projection onto the linear span of δ_1 ; take P_2 to be the orthogonal projection onto the linear span of $\{\delta_2, \delta_3\}$; and so on. Since $\|T\|_{\mathcal{A}} = (\sum_{k=1}^{\infty} s_k(T)^p)^{1/p}$, where

$$s_k(T) = \inf\{\|T - R\|_{\text{op}} : \text{rank}(R) < k\},$$

it is easily checked that $\|P_n\|_{\text{mul}} = 1$. On the other hand, since $\|T\|_{\mathcal{A}} = (\text{trace } |T|^p)^{1/p}$, we have $\|P_n\|_{\mathcal{A}} = n^{1/p}$ for all n . As the (P_n) are pairwise orthogonal idempotents, taking $p_n = u_n = P_n$ yields a SUM configuration. We finish by applying Theorem 2.5. \square

REMARK 2.6. The same argument ought to work for other operator ideals on a large class of Banach spaces, although some hypothesis on the space is necessary in order to produce sufficiently many projections.

In the remainder of this section, we shall give some further examples of how our criterion may be applied.

EXAMPLE 2.7. Let (B_n) be a sequence of Banach algebras, such that for each n the algebra B_n has an identity element 1_n , and suppose that $\|1_n\|_{B_n} \rightarrow \infty$ as $n \rightarrow \infty$. Let \mathcal{A} be the completion of the algebraic direct sum $\bigoplus_n B_n$ in either the c_0 or ℓ^p norms for $1 \leq p < \infty$. When regarded as an element of \mathcal{A} , each 1_n is an idempotent with norm $\|1_n\|_{B_n}$ and multiplier norm 1. Hence on taking $u_n = p_n = 1_n$ we have produced a SUM configuration in \mathcal{A} , and so by Theorem 2.5 \mathcal{A} is not approximately amenable.

In particular, we see that the c_0 -sum of a sequence (B_n) of finite-dimensional amenable Banach algebras need not be approximately amenable (just take each B_n to be \mathbb{C}^n with pointwise multiplication and the ℓ^1 -norm). This should be compared with [8, Example 6.1].

EXAMPLE 2.8 (Correction of [17]). Let S be a Brandt semigroup over a group G with *infinite* index set I , and consider the convolution algebra $\ell^1(S)$. It is claimed in [17, Theorem 1.8] that if G is amenable then $\ell^1(S)$ will be approximately amenable: however, the proof contains an error. (Briefly: in the notation of [17], the algebras $\ell^1(S_F)$ are in general *not* unital, since the norm of the identity element in $\ell^1(S_F)$ grows with $|F|$; hence the appeal to results of [8] is invalid.)

Indeed, $\ell^1(S)$ is *not* approximately amenable. For, let \mathcal{A} be the Banach algebra which consists of all \mathbb{C} -valued $I \times I$ matrices $(a_{ij})_{i,j \in I}$ such that $\sum_{i,j \in I} |a_{ij}| < \infty$, equipped with the usual matrix multiplication as its product. As observed in [5] (see the remarks before Theorem 8), the algebra $\ell^1(S)$ has a 1-dimensional ideal $\mathbb{C}\theta$ (where θ is a formal zero element of S), and the quotient algebra $\ell^1(S)/\mathbb{C}\theta$ is isomorphic to $\mathcal{A} \hat{\otimes} \ell^1(G)$. If we let $\varepsilon : \ell^1(G) \rightarrow \mathbb{C}$ be the augmentation character, then the composite map

$$\ell^1(S) \rightarrow \ell^1(S)/\mathbb{C}\theta \cong \mathcal{A} \hat{\otimes} \ell^1(G) \xrightarrow{\text{id} \otimes \varepsilon} \mathcal{A}$$

is a quotient homomorphism of Banach algebras.

Since approximate amenability is inherited by quotients, if $\ell^1(S)$ were approximately amenable, then \mathcal{A} would be too. This is not the case, because it contains SUM configurations. For instance, let $(F_n)_{n \geq 1}$ be a sequence of pairwise disjoint, finite subsets of I such that $|F_n| \rightarrow \infty$ as $n \rightarrow \infty$. Define u_n to be the $I \times I$ matrix with entries given by

$$(u_n)_{ij} = \begin{cases} 1 & \text{if } i = j \in F_n \\ 0 & \text{otherwise.} \end{cases}$$

Then: (i) $\|u_n\|_{\mathcal{A}} = |F_n|$ for all n ; (ii) the (u_n) are pairwise orthogonal idempotents; and (iii) $\|u_n\|_{\text{mul}} = 1$ for all n . Hence, taking $p_n = u_n$ yields a SUM configuration in \mathcal{A} as required.

REMARK 2.9. After the present work was done, we were informed that a joint paper of Maysami-Sadr and Pourabbas also contains a proof that the preceding example is not approximately amenable: see [25, Theorem 4.5]. This is deduced from a more general result, [25, Theorem 3.4], which we suspect might also be provable using our techniques; however, we have not attempted to do this.

The final two examples of this section are already known not to be approximately amenable. Nevertheless, Theorem 2.5 not only provides alternative proofs, but does so in a unified treatment. In both cases it is convenient to use Lemma 2.4.

EXAMPLE 2.10 (A result of Dales and Loy). Let \mathbb{N}_{\min} denote the semilattice whose underlying set is \mathbb{N} and where the ‘product’ of two elements is defined to be their minimum. Any function $\omega : \mathbb{N} \rightarrow [1, \infty)$ satisfying $\omega(1) = 1$ defines a weight function on \mathbb{N}_{\min} , and the weighted convolution algebra $\ell^1(\mathbb{N}_{\min}, \omega)$ is then isomorphic (via the Gelfand transform) to the **Feinstein algebra** A_ω .

If $\liminf_n \omega(n) < \infty$ then $\ell^1(\mathbb{N}_{\min}, \omega)$ has a bounded approximate identity, and hence by results of [4] (or a direct construction) it is boundedly approximately contractible. On the other hand, in the case where $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$, H. G. Dales and R. J. Loy have shown that $\ell^1(\mathbb{N}_{\min}, \omega)$ is not even approximately amenable (private communication).

The argument of Dales and Loy proceeds by direct calculations, patterned on the arguments of [4]. We can now give an alternative proof. Let e_j denote the point mass concentrated at j , regarded as an element of $\ell^1(\mathbb{N}_{\min}, \omega)$. Since $\lim_j \omega(j) = +\infty$, we can inductively construct a strictly increasing subsequence $(j_n)_{n \geq 1} \subset \mathbb{N}$ such that

$$\text{for each } n \text{ and every } k \geq j_n, \omega(k) \geq \omega(j_n). \quad (\spadesuit)$$

Put $E_n = e_{j_n}$; then clearly the sequence (E_n) is unbounded in $\ell^1(\mathbb{N}_{\min}, \omega)$, and satisfies $E_n E_{n+1} = E_n$ for all n . Moreover, given $a = \sum_i a_i e_i \in \ell^1(\mathbb{N}_{\min}, \omega)$, we have

$$\begin{aligned} \|E_n a\|_{1, \omega} &= \left\| \sum_{i=1}^{j_n-1} a_i e_i + \left(\sum_{k \geq j_n} a_k \right) e_{j_n} \right\| = \sum_{i=1}^{j_n-1} |a_i| \omega(i) + \left| \sum_{k \geq j_n} a_k \right| \omega(j_n) \\ &\leq \sum_{i=1}^{j_n-1} |a_i| \omega(i) + \sum_{k \geq j_n} |a_k| \omega(j_n) \\ &\leq \sum_{i=1}^{j_n-1} |a_i| \omega(i) + \sum_{k \geq j_n} |a_k| \omega(k) = \|a\|_{1, \omega}, \end{aligned}$$

where we used the condition (\spadesuit) at the penultimate step. Thus, $\|E_n\|_{\text{mul}} = 1$ for all n ; and by applying Lemma 2.4 and Theorem 2.5, we conclude that $\ell^1(\mathbb{N}_{\min}, \omega)$ is not approximately amenable.

EXAMPLE 2.11 (A result of P. Lawson). Let X be a Banach space with a normalized unconditional basis (δ_n) . It is standard (see, for instance, [23, Propositions 1.c.6 and 1.c.7]) that X may be equipped with an equivalent norm $\|\cdot\|$ that has the following two properties:

- $\|\delta_n\| = 1$ for all n ;
- for every sequence $(\alpha_n) \subset \mathbb{C}$ such that the series $\sum_n \alpha_n \delta_n$ converges in X , and every sequence $(\gamma_n) \subset \mathbb{C}$ such that $|\gamma_n| \leq |\alpha_n|$ for all n , the series $\sum_n \gamma_n \delta_n$ converges in X

and

$$\left\| \sum_n \gamma_n \delta_n \right\| \leq \left\| \sum_n \alpha_n \delta_n \right\|. \quad (2.6)$$

Equipped with $\|\cdot\|$, X thus becomes a commutative Banach algebra with respect to point-wise product of sequences. Note that the multiplier norm of $\sum_n \alpha_n \delta_n$ is then at most $\sup_n |\alpha_n|$.

If $X = c_0$, it is amenable and hence *a fortiori* approximately amenable. In contrast, Lawson has shown [22] that if X is not isomorphic (as a Banach algebra) to c_0 , then it is not even approximately amenable. His proof, which generalizes arguments from [4], starts by making use of the following observation.

LEMMA 2.12. *Suppose that X is not algebra-isomorphic to c_0 . Then the function $n \mapsto \|\sum_{i=1}^n \delta_i\|$, which by (2.6) is non-decreasing, must be unbounded.*

For sake of completeness, we sketch a proof.

Sketch of proof. Suppose that $\sup_n \|\sum_{i=1}^n \delta_i\| = K < \infty$. We derive a contradiction as follows: given $a \in c_{00}$, let $N = \max\{i : a_i \neq 0\}$, and let M be such that $|a_M| = \|a\|_\infty$. Then (2.6) implies that

$$\|a\|_\infty = \|a_M \delta_M\| \leq \left\| \sum_i a_i \delta_i \right\| \leq \|a\|_\infty \left\| \sum_{i=1}^N \delta_i \right\| \leq K \|a\|_\infty.$$

It follows that the map $a \mapsto \sum_i a_i \delta_i$ extends continuously to a Banach space isomorphism $\theta : c_0 \rightarrow X$, contradicting the assumption on X . \square

In the present context, note that Lemma 2.12 immediately furnishes a sequence satisfying the condition of Lemma 2.4 (just take partial sums of the form $\sum_{i=1}^n \delta_i$). Hence by invoking Lemma 2.4 and Theorem 2.5, we obtain an alternative proof of Lawson's result. (For special cases such as $X = \ell^p$, for $1 \leq p < \infty$, it is simpler to construct an explicit SUM configuration rather than go through Lemma 2.4.)

3 Further applications of our criterion to certain function algebras

We now give some other examples covered by Theorem 2.5, which to the authors' knowledge are new.

Let G be a locally compact group and $L^1(G)$ its group algebra. It has been an open question whether every proper Segal subalgebra of $L^1(G)$, as defined in [27, Ch. 6], fails to be approximately amenable. Recently, H. G. Dales and R. J. Loy have provided various examples of Segal algebras on abelian groups, that fail to be approximately amenable (private communication). It was already known [2] that no proper Segal subalgebra of $L^1(G)$ can be *boundedly* approximately amenable.

Segal subalgebras of $L^1(\mathbb{R}^d)$

THEOREM 3.1. *Let $d \in \mathbb{N}$, and let \mathcal{A} be a proper Segal subalgebra of $L^1(\mathbb{R}^d)$. Then \mathcal{A} is not approximately amenable.*

Proof. It is slightly more convenient for us to take Fourier transforms and work in the Fourier algebra $A(\widehat{\mathbb{R}^d})$. Let $\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow A(\widehat{\mathbb{R}^d})$ be the Fourier transform, normalized so as to be an isometric algebra isomorphism. Then \mathcal{F} maps the Segal subalgebra \mathcal{A} onto an ‘abstract Segal subalgebra’ of $A(\widehat{\mathbb{R}^d})$, and we shall denote this image algebra by \mathcal{FA} (equipped with the norm induced from \mathcal{A} via \mathcal{F}). The Segal algebra condition implies that there exists $C > 0$ such that

$$\|gh\|_{A(\widehat{\mathbb{R}^d})} \leq \|gh\|_{\mathcal{FA}} \leq C\|g\|_{A(\widehat{\mathbb{R}^d})}\|h\|_{\mathcal{FA}} \quad \text{for all } g \in A(\widehat{\mathbb{R}^d}), h \in \mathcal{FA}. \quad (3.1)$$

We shall make crucial use of the **de la Vallée Poussin kernel** for $L^1(\mathbb{R}^d)$, see [16, VI.1.13]. Given $r > 0$, let $b_r : \mathbb{R} \rightarrow [0, 1]$ be the function defined by

$$b_r(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq r; \\ 2 - r^{-1}|x| & \text{if } r \leq |x| \leq 2r; \\ 0 & \text{if } |x| \geq 2r. \end{cases}$$

We then define $V_r \in C_c(\widehat{\mathbb{R}^d}) \subset A(\widehat{\mathbb{R}^d})$ by

$$V_r(\mathbf{x}) = b_r(x_1) \cdots b_r(x_d) \quad (\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d).$$

We note that V_r is equal to 1 on the ‘cube’ $[-r, r]^d$, and vanishes outside the cube $[-2r, 2r]^d$.

It is known that $\|\mathcal{F}^{-1}b_r\|_{L^1(\mathbb{R}^d)} \leq 3$ for all r (this follows by rewriting $\mathcal{F}^{-1}b_r$ as a combination of Féjer kernels), and therefore $\|V_r\|_{A(\widehat{\mathbb{R}^d})} \leq \|b_r\|^d \leq 3^d$, for all r . Therefore (3.1) implies that

$$\|V_r f\|_{\mathcal{FA}} \leq C\|f\|_{\mathcal{FA}} \quad \text{for all } f \in A(\widehat{\mathbb{R}^d}). \quad (3.2)$$

By [27, Proposition 6.2.5], \mathcal{FA} contains all functions in $A(\widehat{\mathbb{R}^d})$ that have compact support; in particular, $V_r \in \mathcal{FA}$ for all r .

LEMMA 3.2. $\|V_r\|_{\mathcal{FA}} \rightarrow \infty$ as $r \rightarrow \infty$.

Proof. Suppose otherwise; then there would exist a constant $K > 0$ and a strictly increasing sequence $r_1 < r_2 < \dots$ in $(0, \infty)$, such that $\|V_{r_n}\|_{\mathcal{FA}} \leq K$ for all n . Hence, for any $f \in A(\widehat{\mathbb{R}^d})$, applying (3.1) yields

$$\|fV_{r_n}\|_{A(\widehat{\mathbb{R}^d})} \leq \|fV_{r_n}\|_{\mathcal{FA}} \leq C\|f\|_{A(\widehat{\mathbb{R}^d})}\|V_{r_n}\|_{\mathcal{FA}} \leq KC\|f\|_{A(\widehat{\mathbb{R}^d})} \quad \text{for all } n.$$

Now, given $f \in C_c(\widehat{\mathbb{R}^d})$, choose n sufficiently large that $\text{supp}(f) \subseteq [-r_n, r_n]^d$. Since $fV_{r_n} = f$, the preceding inequalities imply that

$$\|f\|_{A(\widehat{\mathbb{R}^d})} \leq \|f\|_{\mathcal{FA}} \leq KC\|f\|_{A(\widehat{\mathbb{R}^d})}. \quad (3.3)$$

Since $C_c(\widehat{\mathbb{R}^d})$ is dense in \mathcal{FA} (see [27, Proposition 6.2.8]), it follows from (3.3) that the norms $\|\cdot\|_{\mathcal{FA}}$ and $\|\cdot\|_{A(\widehat{\mathbb{R}^d})}$ are equivalent, and hence that the norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{L^1(\mathbb{R}^d)}$ are equivalent. Since \mathcal{A} is dense in $L^1(\mathbb{R}^d)$ this implies that $\mathcal{A} = L^1(\mathbb{R}^d)$, which is a contradiction. \square

For each n , let $E_n = \mathcal{F}^{-1}(V_{2^n})$. We know by (3.2) that the sequence (V_{2^n}) is bounded in the $\|\cdot\|_{\text{mul}}$ -norm, and Lemma 3.2 tells us that it is unbounded in \mathcal{FA} ; so pulling back by \mathcal{F} we see that (E_n) is unbounded but multiplier-bounded in \mathcal{A} . It remains only to note that since $\mathcal{F}(E_{n+1})$ is equal to 1 on the cube $[-2^{n+1}, 2^{n+1}]^d$, while $\mathcal{F}(E_n)$ is supported inside that cube, we have $E_n E_{n+1} = E_n$ for all n . Then, invoking Lemma 2.4 and Theorem 2.5, we conclude that \mathcal{A} is not approximately amenable. \square

REMARKS 3.3.

- (i) Unlike the examples of the previous section, the sequence $(E_n)_{n \geq 1}$ that was constructed above does not consist of idempotents. This illustrates how it would have been too restrictive to only consider orthogonal idempotents in the definition of a SUM configuration.
- (ii) Essentially the same arguments show that proper Segal subalgebras of the convolution algebra $L^1(\mathbb{T}^d)$, $1 \leq d < \infty$, are not approximately amenable. We have omitted the details; the result had also been observed by R. J. Loy, in response to a preliminary draft of this manuscript (private communication).
- (iii) It seems plausible that a similar argument can be carried out for (abstract Segal subalgebras of) other Fourier algebras. The difficulty would lie in obtaining suitable analogues of the de la Vallée Poussin kernel, in order to allow an appeal to Lemma 2.4.

Little Lipschitz algebras on compact metric spaces

Let (X, d) be a compact, infinite metric space, and fix $\alpha \in (0, 1]$. If $f \in C(X)$ let $\Delta f(x, y) = d(x, y)^{-\alpha}[f(x) - f(y)]$ for $x \neq y$, and put

$$p_\alpha(f) = \sup_{x \neq y} |\Delta f(x, y)|.$$

We briefly recall the definition of the Lipschitz algebras associated to (X, d) and α : further details and proofs can be found in [28]. Recall that $\text{Lip}_\alpha(X, d)$ is the space of all

$f \in C(X)$ such that $p_\alpha(f) < \infty$, equipped with the norm $\|f\|_\alpha = \|f\|_\infty + p_\alpha(f)$; and $\text{lip}_\alpha(X, d)$ is the closed subspace of $\text{Lip}_\alpha(X, d)$ consisting of those $f \in \text{Lip}_\alpha(X, d)$ such that

$$\Delta f(x, y) \rightarrow 0 \text{ as } d(x, y) \rightarrow 0.$$

We equip $\text{lip}_\alpha(X, d)$ with the norm inherited from $\text{Lip}_\alpha(X, d)$; with respect to pointwise product of functions, $\text{lip}_\alpha(X, d)$ is then a unital, commutative Banach algebra, which by results of [1] is known not to be amenable.

THEOREM 3.4. *$\text{lip}_\alpha(X, d)$ is not approximately amenable.*

Our proof is a slightly indirect application of the previous techniques. Since X is infinite and compact, it contains at least one non-isolated point, z_0 say. Let

$$\mathcal{A}_0 = \{f \in \text{lip}_\alpha(X, d) : f(z_0) = 0\};$$

then \mathcal{A}_0 is a maximal ideal in $\text{lip}_\alpha(X, d)$, and the latter may be realized as the unitization of the former. Our aim is to show that \mathcal{A}_0 is not approximately amenable, by constructing a sequence $(E_n) \subset \mathcal{A}_0$ which satisfies the conditions of Lemma 2.4.

Since z_0 is a non-isolated point in X , there exists a sequence $(y_n) \subset X \setminus \{z_0\}$ such that $d(y_n, z_0) \rightarrow 0$. Furthermore, by passing to a subsequence if necessary, we may assume that $d(y_{n+1}, z_0) < d(y_n, z_0)/2$ for all n .

For ease of notation, we put $\delta_n := d(y_n, z_0)$ and let $V_n := \{x \in X : d(x, z_0) \leq \delta_n/2\}$. Also, given a compact subset $S \subseteq X$ and $x \in X$, we write $d(x, S)$ for the distance from x to S , i.e.

$$d(x, S) := \inf\{d(x, y) : y \in S\}.$$

(The infimum is attained, by compactness.)

Define $E_n \in C(X)$ by

$$E_n(x) := \min\{1, 2\delta_n^{-1}d(x, V_n)\}. \quad (3.4)$$

Clearly $0 \leq E_n(x) \leq 1$ for all x , while E_n is zero on V_n . Moreover, if $E_n(x) < 1$ then there exists $u \in V_n$ with $d(x, u) < \delta_n/2$, so that $d(x, z_0) < \delta_n$ by the triangle inequality. Therefore,

$$\text{if } x \in X \text{ and } d(x, z_0) \geq \delta_n \text{ then } E_n(x) = 1. \quad (3.5)$$

In particular, the hypothesis that $\delta_{n+1} < \delta_n/2$ implies that $E_{n+1}(x) = 1$ whenever x lies outside V_n , and it follows that

$$E_n(x)E_{n+1}(x) = E_n(x) \quad \text{for all } x \in X. \quad (3.6)$$

LEMMA 3.5. *Let $x, y \in X$ with $x \neq y$. Then*

$$|\Delta E_n(x, y)| \leq \min\{2\delta_n^{-1}d(x, y)^{1-\alpha}, d(x, y)^{-\alpha}\} \leq 2^\alpha \delta_n^{-\alpha}. \quad (3.7)$$

Proof. Since ΔE_n is an antisymmetric function, it suffices to consider the case $d(x, V_n) \geq d(y, V_n)$. Then, if $d(y, V_n) \geq \delta_n/2$ we have $E_n(x) = E_n(y) = 1$, so that $\Delta E_n(x, y) = 0$.

We thus restrict attention to the case where $d(y, V_n) < \delta_n/2$. For such y , we have $E_n(y) = 2\delta_n^{-1}d(y, V_n)$. Then, since $E_n(x) \leq 2\delta_n^{-1}d(x, V_n)$ for all x , and since $d(x, V_n) \leq d(y, V_n) + d(x, y)$ by the triangle inequality, we have

$$0 \leq \Delta E_n(x, y) \leq \frac{2\delta_n^{-1}d(x, V_n) - 2\delta_n^{-1}d(y, V_n)}{d(x, y)^\alpha} \leq 2\delta_n^{-1}d(x, y)^{1-\alpha}. \quad (3.8a)$$

On the other hand, we have a trivial upper bound

$$0 \leq \Delta E_n(x, y) \leq \frac{1 - 0}{d(x, y)^\alpha} = d(x, y)^{-\alpha}. \quad (3.8b)$$

Combining (3.8a) and (3.8b) gives the first inequality in (3.7). The second one follows by observing that, since $0 < \alpha \leq 1$, the function $t \mapsto \min\{2\delta_n^{-1}t^{1-\alpha}, t^{-\alpha}\}$ attains its maximum at $t = \delta_n/2$. \square

PROPOSITION 3.6. *Let $n \in \mathbb{N}$. Then $E_n \in \mathcal{A}_0$, and:*

- (i) $\delta_n^{-\alpha} \leq p_\alpha(E_n) \leq 2^\alpha \delta_n^{-\alpha}$;
- (ii) *for all $f \in \mathcal{A}_0$, we have $\|E_n f\|_\infty \leq \|f\|_\infty$ and $p_\alpha(E_n f) \leq 3p_\alpha(f)$.*

Proof. The inequalities in (3.7) show that $E_n \in \text{lip}_\alpha(X, d)$, and by construction $E_n(z_0) = 0$. Thus $E_n \in \mathcal{A}_0$. The upper bound in (i) is also immediate from (3.7), while the lower bound follows from the estimate

$$p_\alpha(E_n) \geq \Delta E_n(y_n, z_0) = \frac{2\delta_n^{-1}d(y_n, V_n)}{d(y_n, z_0)^\alpha} \geq 2\delta_n^{-1} \frac{d(y_n, z_0) - \delta_n/2}{d(y_n, z_0)^\alpha} = \delta_n^{-\alpha}.$$

The first estimate in (ii) is trivial. For the second, we argue as follows. If both $d(x, V_n)$ and $d(y, V_n)$ are $\geq \delta_n/2$, then it follows from the definition of E_n , that

$$|\Delta(E_n f)(x, y)| = |\Delta f(x, y)| \leq p_\alpha(f).$$

If $d(x, V_n) < \delta_n/2$, then $d(x, z_0) \leq \delta_n$, and so since $f(z_0) = 0$ we have

$$|f(x)| = d(x, z_0)^\alpha |\Delta f(x, z_0)| \leq \delta_n^\alpha p_\alpha(f).$$

Hence using part (i) yields

$$\begin{aligned} |\Delta(E_n f)(x, y)| &= |[\Delta E_n](x, y) \cdot f(x) + E_n(y) \cdot [\Delta f](x, y)| \\ &\leq p_\alpha(E_n) \cdot \delta_n^\alpha p_\alpha(f) + 1 \cdot p_\alpha(f) \\ &\leq 3p_\alpha(f). \end{aligned}$$

By symmetry, the same argument with x and y interchanged shows that if $d(y, V_n) < \delta_n/2$ then $|\Delta(E_n f)(x, y)| \leq 3p_\alpha(f)$. Thus $p_\alpha(E_n f) \leq 3p_\alpha(f)$, as claimed. \square

Proof of Theorem 3.4. By Proposition 3.6, we can apply Lemma 2.4 and thence Theorem 2.5 to the Banach algebra \mathcal{A}_0 , to conclude that it is not approximately amenable. Since there is an isomorphism of Banach algebras $\mathcal{A}_0^\# \cong \text{lip}_\alpha(X, d)$, it follows from [8, Proposition 2.4] that $\text{lip}_\alpha(X, d)$ is not approximately amenable. \square

4 Approximate amenability of tensor products

It was claimed in [8, Proposition 2.3] that if \mathcal{A} is approximately amenable and has a bounded approximate identity, and \mathcal{B} is amenable, then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is approximately amenable. However, the argument there is incomplete; the correct partial result is as follows.

PROPOSITION 4.1. *Let \mathcal{A} and \mathcal{B} be Banach algebras, where \mathcal{A} is approximately amenable and has a bounded approximate identity, and \mathcal{B} is amenable. Let X be a Banach $\mathcal{A} \hat{\otimes} \mathcal{B}$ -bimodule. Then for every bounded derivation $D : \mathcal{A} \hat{\otimes} \mathcal{B} \rightarrow X^*$, there exists a net $(\xi_\alpha) \subset X^*$ such that $D(a \otimes b) = \lim_\alpha (a \otimes b) \cdot \xi_\alpha - \xi_\alpha \cdot (a \otimes b)$ for all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$.*

Moreover, if \mathcal{A} is furthermore assumed to be boundedly approximately amenable, then

- (i) *there exists $C > 0$ such that the net (ξ_α) may be chosen to satisfy*

$$\|(a \otimes b) \cdot \xi_\alpha - \xi_\alpha \cdot (a \otimes b)\| \leq C \|a\| \|b\| \quad \text{for all } a \in \mathcal{A}, b \in \mathcal{B}; \quad (4.1)$$

- (ii) *$\mathcal{A} \hat{\otimes} \mathcal{B}$ is boundedly approximately amenable.*

Proof. For the first part, we follow the proof of [14, Proposition 5.4]: note that in this argument it is important to first use amenability of \mathcal{B} and only then the approximate amenability of \mathcal{A} , rather than the other way round. If we assume furthermore that \mathcal{A} is boundedly approximately amenable, then inspection of the argument shows that the net (ξ_α) thus produced will satisfy the upper bound claimed in (4.1). Finally, using (4.1) and the definition of the projective tensor product, it is easily checked that $\|w \cdot \xi_\alpha - \xi_\alpha \cdot w\| \leq C \|w\|$ and $D(w) = \lim_\alpha w \cdot \xi_\alpha - \xi_\alpha \cdot w$ for all $w \in \mathcal{A} \hat{\otimes} \mathcal{B}$; thus, every bounded derivation from $\mathcal{A} \hat{\otimes} \mathcal{B}$ to a neo-unital bimodule is the strong-operator limit of a bounded net of inner derivations. By the reasoning of [8, Proposition 2.5], this suffices to show that $\mathcal{A} \hat{\otimes} \mathcal{B}$ is boundedly approximately amenable. \square

We do not know if the tensor product of two (boundedly) approximately amenable Banach algebras is itself (boundedly) approximately amenable. The following results allow us to make some progress.

LEMMA 4.2. *Let \mathcal{A} be approximately amenable and let \mathcal{B} be a finite-dimensional, amenable algebra. Then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is approximately amenable.*

Proof. It suffices to consider derivations $D : \mathcal{A} \hat{\otimes} \mathcal{B} \rightarrow X^*$, where X is a neo-unital $\mathcal{A} \hat{\otimes} \mathcal{B}$ bimodule X . By Proposition 4.1, there exists a net $(\xi_\alpha) \subset X^*$ such that, for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $D(a \otimes b) = \lim_\alpha (a \otimes b) \cdot \xi_\alpha - \xi_\alpha \cdot (a \otimes b)$. But since \mathcal{B} is finite-dimensional, every

element of $\mathcal{A} \widehat{\otimes} \mathcal{B}$ can be written as a *finite* linear combination of elementary tensors, so by linearity we have $D(w) = \lim_{\alpha} w \cdot \zeta_{\alpha} - \zeta_{\alpha} \cdot w$ for every $w \in \mathcal{A} \widehat{\otimes} \mathcal{B}$, and the proof is complete. \square

THEOREM 4.3. *Let \mathcal{A} and \mathcal{B} be Banach algebras, with central bounded approximate identities $(u_i)_{i \in I}$ and $(v_j)_{j \in J}$ respectively. Suppose that there exists a directed family $(\mathcal{B}_{\gamma})_{\gamma \in \Gamma}$ of finite-dimensional, amenable subalgebras of \mathcal{B} , that satisfy the following conditions:*

- (i) $\bigcup_{\gamma} \mathcal{B}_{\gamma}$ is dense in \mathcal{B} ;
- (ii) for each j , there exists $\gamma(j)$ such that $v_j \mathcal{B} = \mathcal{B} v_j \subseteq \mathcal{B}_{\gamma(j)}$.

Then if \mathcal{A} is approximately amenable, so is $\mathcal{A} \widehat{\otimes} \mathcal{B}$.

Proof of theorem. Equip $I \times J$ with the product ordering: then the net $(u_i \otimes v_j)_{i \times j}$ is a central, bounded approximate identity for $\mathcal{A} \widehat{\otimes} \mathcal{B}$, and $(u_i \otimes v_j) \mathcal{B} = \mathcal{B}(u_i \otimes v_j) \subseteq \bigcup_{\gamma} \mathcal{A} \widehat{\otimes} \mathcal{B}_{\gamma}$ for every $(i, j) \in I \times J$. Also, by Lemma 4.2, for each γ the algebra $\mathcal{A} \widehat{\otimes} \mathcal{B}_{\gamma}$ is approximately amenable. Therefore, by [11, Theorem 2.3], $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is pseudo-amenable; since it has a bounded approximate identity, it is approximately amenable, by [12, Proposition 3.2]. \square

EXAMPLE 4.4. In Theorem 4.3 we can take \mathcal{B} to be any of the following examples:

- (i) $\mathcal{B} = C_r^*(G)$ where G is a compact group. In this case the dual object \widehat{G} is discrete, and we may identify \mathcal{B} with the c_0 -sum $\bigoplus_{\sigma \in \widehat{G}} \mathbb{M}_{d(\sigma)}(\mathbb{C})$, where $d(\sigma)$ is the (finite) dimension of the irreducible representation σ . Take Γ to be the set of finite subsets of \widehat{G} , partially ordered by inclusion. For each $F \in \Gamma$ let \mathcal{B}_F be the c_0 -sum $\bigoplus_{\sigma \in F} \mathbb{M}_{d(\sigma)}(\mathbb{C})$, and let v_F be the identity element of the finite-dimensional algebra \mathcal{B}_F . Then $(v_F)_{F \in \Gamma}$ is a central bounded approximate identity in \mathcal{B} , and it is easily verified that the conditions of Theorem 4.3 are satisfied.
- (ii) $\mathcal{B} = \ell^1(\mathbb{N}_{\min})$, the convolution algebra of the semilattice \mathbb{N}_{\min} . Denoting the canonical unit basis of $\ell^1(\mathbb{N}_{\min})$ by $(e_j)_{j \in \mathbb{N}}$, let \mathcal{B}_n be the linear span of $\{e_1, \dots, e_n\}$. Then \mathcal{B}_n is a subalgebra (in fact, an ideal) in \mathcal{A} and is known to be isomorphic to \mathbb{C}^n with pointwise multiplication (so in particular is amenable). Moreover, for each n we have $e_n \mathcal{B} = \mathcal{B}_n = \mathcal{B} e_n$, and $(e_n)_{n \geq 1}$ is a bounded approximate identity for \mathcal{B} . Hence, once again, the conditions of Theorem 4.3 are satisfied.

5 Commutative algebras whose underlying Banach space is reflexive

Any amenable Banach algebra whose underlying Banach space is Hilbertian (i.e. isomorphic to a Hilbert space) must be finite-dimensional: this is well-known and follows from [15, Theorem 2.2]. Various authors have generalized this result to other reflexive Banach

spaces, but only under certain extra hypotheses on the ideal structure of the Banach algebra in question.

In this section we make some partial progress towards corresponding results for bounded approximate amenability. Specifically, we obtain a complete answer in the case of *commutative* Banach algebras. Our main tool is the following result.

PROPOSITION 5.1. *Let \mathcal{A} be a unital, commutative Banach algebra. Suppose that every maximal ideal in \mathcal{A} has an identity element. Then \mathcal{A} is finite-dimensional and isomorphic to \mathbb{C}^N for some N .*

Proof. Let Φ be the character space of \mathcal{A} , equipped with the Gelfand topology. Our first step is to show that Φ is finite; since Φ is compact, it will suffice to show that each point of Φ is isolated (i.e. open and closed).

Let $\varphi \in \Phi$. By assumption, $\ker \varphi$ has an identity element, u_φ say; a small calculation yields

$$(1 - u_\varphi)a = \varphi(a)(1 - u_\varphi) \quad \text{for all } a \in \mathcal{A}. \quad (5.1)$$

Now, if $\chi \in \Phi \setminus \{\varphi\}$ then there exists $b \in \mathcal{A}$ such that $\chi(b) \neq \varphi(b)$. Hence, by (5.1),

$$(1 - \chi(u_\varphi))\chi(b) = \chi((1 - u_\varphi)b) = (1 - \chi(u_\varphi))\varphi(b)$$

so that $\chi(u_\varphi) = 1$. Hence the Gelfand transform of u_φ is zero at φ and 1 at all other points of Φ , which implies that φ is open and closed in Φ as required.

Therefore, Φ is a finite set, consisting of characters $\varphi_1, \dots, \varphi_N$, say. Using the elements $1 - u_{\varphi_j}$, for $1 \leq j \leq N$, it is clear that the Gelfand transform is a surjection onto $C(\Phi) \cong \mathbb{C}^N$. To finish, note that the Jacobson radical of \mathcal{A} , which we shall denote by $\text{Rad}(\mathcal{A})$, is equal to $\bigcap_{j=1}^N \ker \varphi_j$. Put $u := u_{\varphi_1} \cdots u_{\varphi_N} \in \text{Rad}(\mathcal{A})$; then $ux = x$ for all $x \in \text{Rad}(\mathcal{A})$. This is only possible if $\text{Rad}(\mathcal{A}) = 0$, and so the Gelfand transform is injective, which completes the proof. \square

Recall that a Banach algebra \mathcal{A} is a **dual Banach algebra**, if $\mathcal{A} = (\mathcal{A}_*)^*$ for some Banach space \mathcal{A}_* , and the algebra multiplication is separately weak* continuous. We denote the canonical embedding of \mathcal{A}_* into \mathcal{A}^* by ι .

LEMMA 5.2. *Let \mathcal{A} be a dual Banach algebra and be boundedly approximately amenable. Then \mathcal{A} has an identity.*

Proof. From [2, Theorem 2.5] there are nets $(F_i), (G_i) \subseteq \mathcal{A}^{**}$ such that $aF_i \rightarrow a$ and $G_ia \rightarrow a$, for every $a \in \mathcal{A}$, and (F_i) and (G_i) are multiplier bounded. Since $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra, ι^* is an \mathcal{A} bimodule morphism from \mathcal{A}^{**} onto \mathcal{A} , and so $(\iota^*(F_i))$ and $(\iota^*(G_i))$ are multiplier-bounded right and left approximate identities for \mathcal{A} . Therefore, by [2, Theorem 3.3] \mathcal{A} has a bounded approximate identity; taking a weak* cluster point yields an identity. \square

THEOREM 5.3. *Let \mathcal{A} be a commutative, boundedly approximately amenable Banach algebra. If the underlying Banach space of \mathcal{A} is reflexive, then \mathcal{A} is finite-dimensional.*

Proof. By the previous lemma, \mathcal{A} has an identity element $1_{\mathcal{A}}$. By renorming \mathcal{A} if necessary, we may assume that $\|1_{\mathcal{A}}\| = 1$, i.e. that \mathcal{A} is unital.

Now let M be a maximal ideal in \mathcal{A} . Clearly M is reflexive; moreover, since $M^{\#} \cong \mathcal{A}$ and \mathcal{A} is boundedly approximately amenable, M is also boundedly approximately amenable (see [2, Proposition 2.4]), and hence (by Lemma 5.2) it has an identity element. Applying Proposition 5.1 completes the proof. \square

6 Bounded approximate amenability of $A(G)^{**}$ and $A(G)$

For G a locally compact group, $L^1(G)$ denotes the group algebra; $A(G)$ and $B(G)$ denote, respectively, the **Fourier** and **Fourier-Stieltjes** algebra of G , as defined by Eymard [6]. Amenability and approximate amenability of $L^1(G)^{**}$ were studied in [9] and [8] respectively, while amenability of $A(G)^{**}$ was characterized in [13] and [21]. Here we characterize bounded approximate amenability of $A(G)^{**}$.

Throughout this section, the second dual algebras are equipped with the first Arens product, which we denote by \square . For further details on Arens products, see [3, §3.3] or [26, §1.4].

PROPOSITION 6.1. *Let \mathcal{A} be a commutative Banach algebra. If \mathcal{A}^{**} is boundedly approximately amenable, then \mathcal{A} has a bounded approximate identity. Moreover, \mathcal{A}^{**} has an identity.*

Proof. From Theorem 2.6 of [2] there are nets $(M_i) \subseteq (\mathcal{A}^{**} \widehat{\otimes} \mathcal{A}^{**})^{**}$, $(F_i) \subseteq \mathcal{A}^{****}$, $(G_i) \subseteq \mathcal{A}^{****}$ and $K > 0$, such that

$$\begin{aligned} \|u \cdot M_i - M_i \cdot u - u \otimes G_i + F_i \otimes u\| &\leq K\|u\|, \\ u \cdot F_i &\rightarrow u, \quad \|u \cdot F_i\| \leq K\|u\|, \\ G_i \cdot u &\rightarrow u, \quad \|G_i \cdot u\| \leq K\|u\|, \quad (u \in \mathcal{A}^{**}). \end{aligned} \tag{6.1}$$

Let ι be the canonical embedding of \mathcal{A}^* in \mathcal{A}^{****} . Then $P = \iota^*$ is an \mathcal{A} -module morphism from \mathcal{A}^{****} onto \mathcal{A}^{**} . Now, by using weak*-continuity of the first Arens product in the first variable, we can show that P is a right \mathcal{A}^{**} -morphism. In fact, take $F \in \mathcal{A}^{****}$ and $m \in \mathcal{A}^{**}$. Let (f_i) be a net of elements of \mathcal{A}^{**} converging weak* to F . Then

$$P(F \cdot m) = \text{weak}^* \lim_i P(f_i m) = \text{weak}^* \lim_i f_i m = \text{weak}^* \lim_i P(f_i) m = P(F) m.$$

Hence by applying P to the third equation in (6.1) we see that \mathcal{A}^{**} has a multiplier-bounded left approximate identity, (f_{β}) , say. From (6.1) again, for $u = f_{\beta}$ we have

$$\|(f_{\beta} \cdot M_i - M_i f_{\beta} - f_{\beta} \otimes G_i + F_i \otimes f_{\beta}) \cdot F_{\alpha}\| \leq K^2 \|f_{\beta}\|, \text{ for every } \alpha, \beta \text{ and } i. \tag{6.2}$$

By using the triangle inequality, and left-multiplier-boundedness of the set (f_{β}) , from (6.2) we have

$$\begin{aligned} \|f_{\beta}\| \|G_i \square F_{\alpha}\| &\leq K^2 \|f_{\beta}\| + \|f_{\beta} \cdot M_i \cdot F_{\alpha}\| + \|M_i \cdot (f_{\beta} F_{\alpha})\| + \|F_i\| \|f_{\beta} F_{\alpha}\| \\ &\leq K^2 \|f_{\beta}\| + K \|M_i \cdot F_{\alpha}\| + K \|M_i\| \|F_{\alpha}\| + K \|F_i\| \|F_{\alpha}\|, \end{aligned} \tag{6.3}$$

for all α, β and i .

Hence,

$$\|G_i \square F_\alpha\| \leq K^2 + \frac{K}{\|f_\beta\|} (\|M_i \cdot F_\alpha\| + \|M_i\| \|F_\alpha\| + \|F_i\| \|F_\alpha\|), \text{ for all } \alpha, \beta \text{ and } i. \quad (6.4)$$

Now two cases may occur.

Case 1: (f_β) is unbounded. In this case, from (6.4) we have

$$\|G_i \square F_\alpha\| \leq K^2, \quad \text{for all } i \text{ and } \alpha.$$

For every $f \in \mathcal{A}$, noting that $f \cdot G_i = G_i \cdot f$ as \mathcal{A} is commutative, we have $f \cdot G_i \square F_\alpha \rightarrow f$ as α and i tend to infinity. Hence $f \cdot P(G_i \square F_\alpha) \rightarrow f$. Taking E to be any weak*-cluster point of $(P(G_i \square F_\alpha))_{i,\alpha}$, it follows that E is a right identity for \mathcal{A}^{**} , and hence \mathcal{A} has a bounded approximate identity.

Case 2: the net (f_β) is bounded. Then we may argue as in case 1, to conclude that \mathcal{A} has a bounded approximate identity.

For the final part: let (e_j) be a bounded approximate identity for \mathcal{A} , and let E be a weak*-cluster point of (\hat{e}_j) . Then E is a right identity for \mathcal{A}^{**} . On the other hand, since \mathcal{A}^{**} is boundedly approximately amenable, it is in particular approximately amenable. By [8, Lemma 2.2], it therefore has a left approximate identity, say (n_j) , and we obtain

$$E \square F = \lim_j n_j \square E \square F = \lim_j n_j \square F = F, \quad \text{for every } F \in \mathcal{A}^{**}. \quad (6.5)$$

Thus E is also a left identity for \mathcal{A}^{**} . □

The proof of the following theorem uses, in part, an idea from the proof of [8, Theorem 3.3], but also some properties specific to the Fourier algebra. Proposition 6.1 plays an important role.

THEOREM 6.2. *Let G be a locally compact group. Then $A(G)^{**}$ is boundedly approximately amenable if and only if G is finite.*

Proof. Sufficiency is trivial. Suppose, then, that $A(G)^{**}$ is boundedly approximately amenable.

Let n be a topological invariant mean on $VN(G) = A(G)^*$, that is, a state which satisfies $f \cdot n = \langle f, I \rangle n$ for all $f \in A(G)$, where $I \in VN(G)$ is the identity operator. Then $r \square n = n$ for every state $r \in VN(G)^*$, by weak*-continuity of \square in the first variable. Hence $J_n = n \square A(G)^{**}$ is a closed, complemented, two-sided ideal in $A(G)^{**}$ and so, by [8, Corollary 2.4], J_n has a right approximate identity, (p_i) say.

If m is any topological invariant mean on $VN(G)$, then $m = n \square m \in J_n$. Since $p_i \in J_n$, $m \square p_i = p_i$, and therefore

$$m = \lim m \square p_i = \lim p_i, \quad (6.6)$$

showing that $VN(G)$ has a unique topological invariant mean. Hence by [20, Corollary 4.11], G is discrete.

On the other hand, since $A(G)^{**}$ is boundedly approximately amenable, it follows from the preceding proposition that it has a two-sided identity. By [19, Proposition 3.2(b)], G must be compact. Since we have also shown that G is discrete, it follows that G is finite. \square

We now examine **essential amenability** ([8, §1]) of $A(G)^{**}$. It suffices to know the following two results: if A is an essentially amenable Banach algebra and there exists a continuous algebra homomorphism of A onto another Banach algebra B , then by [8, Proposition 2.2], B is essentially amenable; and if A is an essentially amenable Banach algebra with an identity element, then A is amenable.

Recall that $UCB(\widehat{G})$ is defined to be the closure of $A(G) \cdot VN(G)$, where $VN(G)$ is identified with the dual of $A(G)$ and the action of $A(G)$ on $VN(G)$ is the canonical action of an algebra on its dual. The reduced group C^* -algebra $C_r^*(G)$ is contained in $UCB(\widehat{G})$ (see [18, Proposition 4.4] for a proof).

For the proof of the theorem which follows, it is convenient to work with the notion of a ‘left-introverted’ subspace of $VN(G)$ (for the definition, see [18, §5], for example). Examples of left-introverted subspaces of $VN(G)$ are: $VN(G)$ itself; $UCB(\widehat{G})$; and the reduced C^* -algebra $C_r^*(G)$ ([18, Proposition 5.2]). The relevance of these subspaces to the present study of $A(G)^{**}$ arises from the following lemma, which follows from the discussion on [21, p. 177].

LEMMA 6.3. *Let $Y \subseteq X$ be two left-introverted subspaces of $VN(G)$. Then the duals Y^* and X^* can both be equipped with Arens-type products, with respect to which the canonical quotient maps $VN(G)^* \rightarrow X^*$ and $X^* \rightarrow Y^*$ are algebra homomorphisms.*

In particular, there are quotient homomorphisms $A(G)^{**} \rightarrow UCB(\widehat{G})^*$ and $UCB(\widehat{G})^* \rightarrow C_r^*(G)^*$. Moreover, the algebra $C_r^*(G)^*$ is naturally embedded as a closed subalgebra of $B(G)$ ([18, Proposition 5.3]), and is equal to $B(G)$ if and only if G is amenable.

THEOREM 6.4. *Let G be an amenable group. Then either of the Banach algebras $A(G)^{**}$ or $UCB(\widehat{G})^*$ is essentially amenable if and only if G is finite.*

Proof. If G is finite, then $A(G)^{**} = UCB(\widehat{G})^* = A(G)$ is amenable, and in particular essentially amenable. Conversely, if either $A(G)^{**}$ or $UCB(\widehat{G})^*$ is essentially amenable, then so is $C_r^*(G)^*$. Since G is amenable, it follows that $B(G) = C_r^*(G)^*$ is also essentially amenable, and hence amenable as it has an identity. Corollary 2.4 of [7] then implies that G is compact, so that $A(G)$ has an identity element. Therefore $UCB(\widehat{G}) = A(G) \cdot VN(G) = VN(G)$, and so $UCB(\widehat{G})^* = A(G)^{**}$ has an identity and is essentially amenable, hence is amenable. By the results of [13] (or Theorem 6.2 above), we conclude that G is finite. \square

For a general locally compact group G , there is still no complete characterization of the bounded approximate amenability of $A(G)$. However, in the special case where G has an open abelian subgroup (in particular, if G is discrete), we can give a complete answer.

THEOREM 6.5. *Suppose that G has an open abelian subgroup. Then $A(G)$ is boundedly approximately amenable, if and only if G is amenable.*

Proof. Suppose that $A(G)$ is boundedly approximately amenable. Then from [2, Proposition 3.6] it follows that every multiplier-bounded subset of $A(G)$ is norm bounded. This is equivalent to saying that the multiplier norm on $A(G)$ is equivalent to its norm. Hence the amenability of G follows from [24, Theorem 1]. The converse follows from a direct application of [11, Corollary 3.2(ii)]. \square

7 Concluding remarks

We have seen that many of the existing ‘non-amenability’ results for Banach algebras arising in abstract harmonic analysis and operator theory admit extensions to the (boundedly) approximately amenable setting; however, in some of these cases, the proofs require genuinely new arguments rather than mere extension of the existing ones.

The technique presented in Section 2 seems deserving of further exploration. Several of the technical hypotheses can perhaps be weakened, although it is unclear if doing so would give a substantially stronger result. The applications presented in Sections 2 and 3 are intended to be illustrative, rather than exhaustive, and we hope they will stimulate further developments.

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